Finiteness theorems for nonnegatively curved vector bundles

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Abstract

We prove several finiteness theorems for the normal bundles to souls in nonnegatively curved manifolds. More generally, we obtain finiteness results for open Riemannian manifolds whose topology is concentrated on compact domains of "bounded geometry".

1 Introduction

Much of the recent work in Riemannian geometry was centered around finiteness and precompactness theorems for various classes of Riemannian manifolds. Some versions of precompactness results typically work for compact domains in Riemannian manifolds. The main point of the present paper is that one can sometimes get diffeomorphism finiteness for ambient Riemannian manifolds provided their topology is concentrated on a compact domains of "bounded geometry". We postpone the discussion of our main technical results till section 5 and concentrate on applications to nonnegative curvature.

Recall that according to the soul theorem of J. Cheeger and D. Gromoll a complete open manifold of nonnegative sectional curvature is diffeomorphic to the total space of the normal bundle of a compact totally geodesic submanifold which is called the soul. One of the harder questions in the subject is what kind of normal bundles can occur. See [Che73, Rig78, Yan95, GZ99, GZ] for examples of open nonnegatively curved manifolds, and [ÖW94, BK00b, BK00a] for known obstructions. Here is our first result.

Theorem 1.1. *Given a closed Riemannian manifold S with* $\sec(S) \geq 0$*, and positive D, r, v, n, there exists a finite collection of vector bundles over S such that, for every complete open Riemannian n-manifolds N with* $\sec(N) \geq 0$ *and an isometric embedding $e\colon S \to N$ of S onto a soul of N the normal bundle ν_e is isomorphic to a bundle of the collection provided e is homotopic to a map f such that* $\operatorname{diam}(f(S)) \leq D$ *and* $\operatorname{vol}_N(B(p,r)) \geq v$ *for some $p \in f(S)$.*

There is also a "variable base version" of 1.1. We say that two vector bundles ξ, ξ' over different bases B, B' are *topologically equivalent* if there is a homeomorphism $h\colon B' \to B$ such that $h^{\#}\xi$ is isomorphic to ξ'. Moreover, if h is a diffeomorphism, we say that ξ and ξ' are *smoothly equivalent*. For bundles over manifolds of dimension ≤ 3, topological equivalence implies smooth equivalence because in this case any homeomorphism is homotopic to a (nearby) diffeomorphism [Mun60, Moi77]. The diffeomorphism type of the total space of a vector bundle (of positive rank over a closed manifold) is determined up to finite ambiguity by the topological equivalence class of the bundle (see [HM74, KS77] if dimension of the total space is ≥ 5 and [Mun60, Moi77] otherwise). In the appendix we discuss to what extent the total space determines a vector bundle and give an example of infinitely many pairwise topologically nonequivalent vector bundles over a closed manifold with diffeomorphic total spaces.

The following result can be thought of as generalizations of the Grove-Petersen-Wu finiteness theorem.

Theorem 1.2. *Given positive D, r, v, v', n, there exists a finite collection of vector bundles such that, for any open complete Riemannian n-manifold N with* $\sec(N) \geq 0$ *and a soul $S \subset N$, the normal bundle to the soul is topologically equivalent to a bundle of the collection provided* $\mathrm{vol}(S) \geq v'$ *and the inclusion $S \hookrightarrow N$ is homotopic to a map f such that* $\mathrm{diam}(f(S)) \leq D$ *and* $\mathrm{vol}_N(B(p,r)) \geq v$ *for some $p \in f(S)$.*

We suspect that in 1.2 the lower volume bound on the soul follows from the rest of the assumptions, and thus can be omitted. For example, it follows from [HK78] that lower volume bound for a soul S comes from a lower volume bound on an ambient manifold N, that is $\mathrm{vol}_N B(p,r) \geq v$ implies $\mathrm{vol}(S) \geq v'$ provided the distance from p to S is uniformly bounded. (The latter can be also forced by purely topological assumptions on S as we show in 6.5.) Thus we deduce the following.

Corollary 1.3. *Given positive D, r, v, n, there exists a finite collection of vector bundles such that, for any open complete Riemannian n-manifold N with* $\sec(N) \geq 0$ *and a soul $S \subset N$, the normal bundle to the soul is topologically equivalent to a bundle of the collection provided* $\mathrm{diam}(S) \leq D$ *and* $\mathrm{vol}_N(B(p,r)) \geq v$ *for some $p \in N$ such that the distance from S to p is uniformly bounded.*

The finiteness of homeomorphism types of total spaces in 1.3 can be easily obtained from the parametrized version of Perelman's Stability theorem [Per91]

and the regularity properties of the distance function; however the conclusion of 1.3 is strictly stronger (cf. A.1).

There is a version of the above corollary for totally geodesic submanifolds in Riemannian manifolds with lower sectional curvature bound. Another version works when ambient manifolds have lower bound on Ricci curvature and injectivity radius.

Perelman proved in [Per94] that the distance nonincreasing retraction onto the soul introduced in [Sha77] is a $C^{1,1}$-Riemannian submersion. We observe that a local bound on the vertical curvatures of this submersion gives a lower volume bound for the ambient nonnegatively curved manifold. In particular, we deduce the following.

Corollary 1.4. *Given positive n, r, K, and a closed nonnegatively curved Riemannian manifold S, there is a finite collection of vector bundles over S such that, for any open complete Riemannian n-manifold N with $\sec(N) \geq 0$ and an isometric embedding $e\colon S \to N$ of S onto a soul of N, the normal bundle ν_e is isomorphic to a bundle of the collection provided $\operatorname{diam}(S) \leq D$ and there is a point $p \in e(S)$ such that all the vertical curvatures at the points of $B(p,r)$ are bounded above by K.*

Note that 1.4 generalizes the main result of [GW98] (cf. [Tap99]) where the same statement is proved for a soul isometric to the round sphere. Again, there is a "variable base" version of 1.4 when souls vary in the Grove-Petersen-Wu class. Similarly, 1.4 holds when the soul varies in Cheeger-Andersen class [AC92]: $\operatorname{diam}(S) \leq D$ and $\operatorname{injrad}(S) \geq i_0$ where the conclusion of finiteness up to topological equivalence gets improved to the finiteness up to smooth equivalence.

There is a counterpart of 1.2 in nonpositive curvature. Let $\sec(N) \leq 0$ and $e\colon M \to N$ be a totally geodesic embedding which is a homotopy equivalence. Then the orthogonal projection $N \to e(M)$ is distance nonincreasing. Moreover, $\operatorname{inj}(N) = \operatorname{inj}(M)$ and we obtain the following corollary (which also has a "fixed base" version).

Corollary 1.5. *Given positive D, r, v, n, K, there exists a finite collection of vector bundles such that, for any totally geodesic embedding $e\colon M \to N$ of a closed Riemannian manifold M into an open complete Riemannian n-manifold N with $\sec(N) \leq 0$, the normal bundle ν_e is topologically equivalent to a bundle of the collection provided $\sec(M) \geq -1$, $\operatorname{vol}(M) \geq v$ and e is a homotopy equivalence homotopic to a map f with $\operatorname{diam}(f(M)) \leq D$ such that the sectional curvature at any point of the r-neighborhood of $f(M)$ is $\geq -K$.*

Normal bundles to totally geodesic embeddings in nonpositively curved manifolds can be fairly arbitrary as the following example shows. M. Anderson proved that the total space $E(\xi)$ of any vector bundle ξ over a closed nonpositively curved manifold M carries a complete metric with $-1 \leq \sec \leq 0$ [And87]. Let M be a closed locally symmetric manifold of nonpositive curvature and $rank(M) \geq 2$ such that no finite cover of M splits as a Riemannian product. Let ξ be an orientable vector bundle over M with nonzero Euler class. Then according to [SY97, p326] the zero section $M \to E(\xi)$ is homotopic to a harmonic map which, by the harmonic map superrigidity [MSY93], is a totally geodesic embedding (after rescaling the metric on $E(\xi)$). The normal bundle to this totally geodesic embedding is stably isomorphic to ξ, and furthermore it has the same Euler class as ξ.

One may wonder when there are *infinitely* many vector bundles of rank m over a given base M. For example, if $m \geq \dim(M)$, this happens whenever M has nonzero Betti number in a dimension divisible by 4 (e.g. if M is a closed orientable manifold of dimension divisible by 4). The reason is that the Pontrjagin character defines an isomorphism of $\oplus_{i>0} H^{4i}(M, \mathbb{Q})$ and $\widetilde{K}(M) \otimes \mathbb{Q}$ where $\widetilde{K}(M)$ is the group of stable equivalence classes of vector bundles over M. Furthermore, the Euler class defines a one-to-one correspondence between the set of isomorphism classes of oriented rank 2 bundles over M and $H^2(M, \mathbb{Z})$. Also, if M is a closed, orientable, and $2n$-dimensional, then there are infinitely many rank $2n$ bundles over M obtained as pullbacks of TS^{2n} via maps $M \to S^{2n}$ of nonzero degree.

The structure of the paper is as follows. Section 2 reviews some well-known results on homotopy count of maps in equicontinuous families. Section 3 discusses local versions of precompactness theorems in [AC92] and [Per91]. The 4th section provides a background in characteristic classes and related invariants of maps. Main technical results are proved in section 5. In section 6 we prove applications to nonnegatively/nonpositively curved manifolds. In the appendix we explain to what extent a vector bundle is determined by its total space.

We are grateful to M. Anderson for an illuminating communication on the local version of [AC92] and to S. Weinberger for the idea of A.1. The first author is thankful to A. Nicas and I. Hambleton for several helpful discussions on self-equivalences of manifolds. The second author is grateful to Kris Tapp for bringing to his attention the idea of bounding homotopy types of maps using equicontinuity and for many helpful conversations on nonnegatively curved manifolds.

2 Equicontinuity and homotopy count of maps

Definition 2.1. *A family of maps of metric spaces $f_\alpha\colon X_\alpha \to Y_\alpha$ is called ϵ-equicontinuous if there exist $\delta > 0$ such that $d_{Y_\alpha}(f_\alpha(x), f_\alpha(x')) < \epsilon$, for any α and any $x, x' \in X_\alpha$ with $d_{X_\alpha}(x, x') < \delta$. A family f_α is called equicontinuous if it is ϵ-equicontinuous for every ϵ. A family f_α is called almost equicontinuous if for any ϵ there exists a finite subset $S_\epsilon \subset \{f_\alpha\}$ whose complement is ϵ-equicontinuous.*

Example 2.2. Assume $f_\alpha\colon X_\alpha \to Y_\alpha$ is a family of maps of metric spaces.

(1) If each f_α is (α, L)-Hölder (i.e. $d_{Y_\alpha}(f_\alpha(x), f(x')) \le L d_{X_\alpha}(x, x')^\alpha$), then $\{f_\alpha\}$ is equicontinuous.

(2) If each f_α is an ϵ_α-Hausdorff approximation (or more generally, if f_α satisfies

$$|d_{Y_\alpha}(f_\alpha(x), f_\alpha(x')) - d_{X_\alpha}(x, x')| \le \epsilon_\alpha$$

for any $x, x' \in X_\alpha$) and $\epsilon_\alpha \to 0$ then, $\{f_\alpha\}$ is almost equicontinuous.

(3) If $\{f_\alpha\}$ is almost equicontinuous and g_α is ϵ_α-close to f_α with $\epsilon_\alpha \to 0$, then $\{g_\alpha\}$ is almost equicontinuous.

(4) If $\{f_\alpha\}$ where $f_\alpha\colon X_\alpha \to Y_\alpha$ is almost equicontinuous and $\{g_\alpha\}$ with $g_\alpha\colon Y_\alpha \to Z_\alpha$ is almost equicontinuous, then $\{g_\alpha \circ f_\alpha\}$ is almost equicontinuous.

The importance of the following result in Riemannian geometry was first observed by M. Gromov [Gro81].

Proposition 2.3. *Let Y be a compact metric space such that there exists an $\epsilon = \epsilon(Y)$ with the property that any two 4ϵ-close continuous maps of a compact metric space into Y are homotopic. Then, given a compact metric space X, any ϵ-equicontinuous family of maps $f_\alpha\colon X \to Y$ falls into finitely many homotopy classes.*

Proof. Fix a $\delta > 0$ such that $d_{Y_\alpha}(f_\alpha(x), f_\alpha(x')) < \epsilon$, for any k and any $x, x' \in X_\alpha$ with $d_{X_\alpha}(x, x') < \delta$. Find a finite $\delta/2$-net N_X in X and a finite ϵ-net N_Y in Y. Any map $f\colon X \to Y$ produces a (nonunique) map $\hat{f}\colon N_X \to N_Y$ defined so that $\hat{f}(x)$ is a point of N_Y whose distance to $f(x)$ is $\le \epsilon$. Now if f and g are ϵ-equicontinuous maps with $\hat{f} = \hat{g}$, then f and g are 4ϵ-close, hence homotopic. In particular, $\{f_\alpha\}$ fall into at most $\mathrm{card}(N_Y)^{\mathrm{card}(N_X)}$ homotopy classes. □

Remark 2.4. Such an $\epsilon(Y)$ exists if, for example, the compact metric spaces X and Y are separable, finite-dimensional ANR [Pet93]. Note that for compact, separable, finite-dimensional metric spaces being ANR is equivalent to being locally contractible [Bor67, V.10.4]; any such space is homotopy equivalent to a finite cell complex [Wes77].

3 Local convergence results

In this section we discuss local versions of the C^{α}-precompactness theorem of M. Anderson and J. Cheeger [AC92] and Perelman's stability theorem [Per91]. The results provide sufficient conditions under which a sequence of compact domains in Riemannian manifolds has uniformly bounded geometry in the sense defined below. In fact, the theorems in [AC92] and [Per91] are stated in a local form so we just give details needed for "compact domains version".

Let U_α be a family of compact domains (i.e. compact codimension zero topological submanifolds) of Riemannian n-manifolds N_α. We say that $\{U_\alpha\}$ has *uniformly bounded geometry* if any sequence of domains in the family has a subsequence $\{U_k\}$ such that there exists a metric space V, and homeomorphisms $h_k\colon V \to V_k$ of V onto compact domains $V_k \supset U_k$ such that both $\{h_k\}$ and $\{h_k^{-1}\}$ are almost equicontinuous. In case $\partial U_k = \emptyset$, we necessarily have $U_k = V_k = N_k$.

Throughout the paper we always denote the closed ϵ-neighborhood of a subspace S by S^ϵ.

Theorem 3.1. **[AC92]** *Given $\epsilon > 0$, let U_k be a sequence of compact domains with smooth boundaries in Riemannian n-manifolds N_k such that the closed ϵ-neighborhood U_k^ϵ of U_k is compact. Assume that for some positive H, V, i_0, the following holds:* $\mathrm{Ric}(U_k^\epsilon) \geq -(n-1)H$, $\mathrm{vol}(U_k^\epsilon) \leq V$, *and* $\mathrm{inj}_{N_k}(x) \geq i_0$ *for any $x \in U_k^{\epsilon/2}$. Then, after passing to a subsequence, there are compact domains V_k with $U_k \subset V_k \subset U_k^{\epsilon/2}$, a manifold V, and C^∞-diffeomorphisms $h_k\colon V \to V_k$ such that the pullback metrics $h_k^* g_k$ converge in a C^α-topology to a C^α-Riemannian metric on the interior of V.*

Proof. For reader's convenience we review the argument in [AC92] emphasizing its local nature. It is proved in [AC92, pp269–270] that any domain $U_k^{\epsilon/2}$ as above has an atlas of harmonic coordinate charts $F_\nu\colon B(x, r_h) \to \mathbb{R}^n$ where $B(x, r_h)$ is a metric ball at $x \in U_k^{\epsilon/2}$ whose radius $r_h \leq \epsilon/10$ depends only on the initial data. Further, the metric tensor coefficients in the charts F_ν are

controlled in C^{α} topology. An elliptic estimate then shows that the transition functions $F_{\mu} \circ F_{\nu}^{-1}$ are controlled in $C^{1,\alpha}$ topology. All these results are stated and proved locally.

Next, the relative volume comparison implies that one can choose a finite subatlas so that there is a uniform bound on the multiplicities of intersections of the coordinate charts and the balls $B(x, r_h/2)$ still cover $U_k^{\epsilon/2}$ (this argument involves only small balls and hence is local). The lower injectivity radius bound gives a lower bound for the volume of any small ball that depends only on the radius of the ball [Cro84]. This, together with a upper bound on $\mathrm{vol}(U_k^{\epsilon})$, implies an upper bound on the number of coordinate charts.

Finally, following Cheeger's thesis (as outlined in [AC92, pp266–267]) one can "glue the charts together" which proves the theorem. Alternatively, one can follow (almost word by word) the argument in [And89, pp464–466] where a "compact domain version" of Cheeger-Gromov convergence theorem is proved. □

Remark 3.2. There are many other convergence theorems, notably those involving integral curvature bounds (see [Pet97]). For example, Hiroshima [Hir95] generalized [AC92] replacing a lower Ricci curvature bound by an integral bound on an eigenvalue of the Ricci curvature. Hiroshima's proof is given for complete manifolds; however, a local version of [Hir95] is likely to hold. We leave this matter for an interested reader to clarify.

Before starting the proof of theorem 3.5, we need a local version of Packing Lemma that ensures Gromov-Hausdorff convergence.

We say that a metric space (X, d) is *locally interior* if for any point $x \in X$ there exists an $\epsilon > 0$ such that for any $y, z \in B(x, \epsilon)$ we have $d(y, z) = \inf_{\gamma} L(\gamma)$ where the infimum is taken over all paths γ connecting y and z. For example, all Riemannian manifolds are locally interior.

Remark 3.3. Notice that for locally compact metric spaces the property of being locally interior is easily seen to be equivalent to the following one. For any point $x \in X$ there exists an $\epsilon > 0$ such that for any $y, z \in B(x, \epsilon)$ there exists a sequence $p_n \in B(x, 2\epsilon)$ such that $d(p_n, y) \to d(y, z)/2$ and $d(p_n, z) \to d(y, z)/2$.

Here is how locally interior spaces arise in this paper. Let V_k be a sequence of compact domains in Riemannian n-manifolds. Equip V_k with induced Riemannian metrics and assume that V_k converges in Gromov-Hausdorff topology to a compact metric space V. Consider $f_k \colon V_k \to \mathbb{R}$ defined by $f_i(x) =$

$dist(x, \partial V_i)$. Then each f_i is 1-Lipschitz and by Arzela-Ascoli Theorem this sequence converges to 1-Lipschitz function $f\colon V \to \mathbb{R}$. We call the open set $\{x \in U\colon f(x) > 0\}$ the *interior* of U. Then it is easy to show that the interior of U is a locally interior space.

Lemma 3.4. *Given $\epsilon > 0$, let U_k be a sequence of compact connected domains with smooth boundaries in Riemannian n-manifolds N_k such that the closed ϵ-neighborhood U_k^ϵ of U_k is compact. Assume that for some positive H, V, v_0, $r_0 < \epsilon/10$, the following holds:* $\mathrm{Ric}(U_k^\epsilon) \geq -(n-1)H$, $\mathrm{vol}(U_k^\epsilon) \leq V$, *and* $\mathrm{vol}(B(x, r_0)) \geq v_0$ *for any $x \in U_k^{\epsilon/2}$. Then, after passing to a subsequence, the compact domains $U_k^{\epsilon/2}$ converge in Gromov-Hausdorff topology to a compact metric space U whose interior is a locally interior metric space.*

Proof. Take an arbitrary $r < r_0$. To prove precompactness in Gromov-Hausdorff topology it is enough to show that the number of elements in a maximal r- net in $U_k^{\epsilon/2}$ is bounded above by some $N(r)$ independent of k.

Fix a maximal r-separated nets N_k in $U_k^{\epsilon/2}$ so that r-balls with centers in N_k are disjoint and $2r$-balls cover U_k^ϵ. The relative volume comparison gives a uniform lower bound for the volume of the r-ball centered at any point of $U_k^{\epsilon/2}$; say $\mathrm{vol}(B(x, r) \geq v$. Then $\# N_k \leq V/v$ and $U_k^{\epsilon/2}$ converge in the Gromov-Hausdorff topology to a compact metric space U. As we explained above the interior of U is necessarily locally interior. □

Theorem 3.5. **[Per91]** *Given $\epsilon > 0$, let U_k be a sequence of compact connected domains with smooth boundaries in Riemannian n-manifolds N_k such that the closed ϵ-neighborhood of U_k, denoted by U_k^ϵ is compact. Assume that for some positive K, V, v, $r < \epsilon/10$,* $\sec(U_k^\epsilon) \geq -K$, $\mathrm{vol}(U_k^\epsilon) \leq V$, *and* $\mathrm{vol}(B(x, r)) \geq v$ *for any $x \in U_k^{\epsilon/2}$. Then, after passing to a subsequence, there are compact domains V_k with $U_k^{\epsilon/4} \subset V_k \subset U_k^{\epsilon/2}$, a manifold V, and homeomorphisms $h_k\colon V \to V_k$ which are ϵ_k-Hausdorff approximations with $\epsilon_k \to 0$ as $k \to \infty$.*

Proof. By 3.4 $U_k^{\epsilon/2}$ subconverges in the Gromov-Hausdorff topology to a compact metric space U whose interior $int(U)$ is a locally interior metric space. We are in position to apply Perelman's stability theorem [Per91] which asserts that $int(U)$ is a topological manifold, and moreover, any compact subset of $int(U)$ lies in a compact domain $V \subset int(U)$ such that there are topological embedding $h_k\colon V \to U_k^{\epsilon/2}$. Furthermore, h_k induce Hausdorff approximations

which become arbitrary close to the given Hausdorff approximations between U and $U_k^{\epsilon/2}$. Choosing V large enough, one can ensure that $h_k(V) \supset U_k^{\epsilon/4}$ as promised. $\square$

Remark 3.6. Let U_k be a sequence of compact domains with smooth boundaries in Riemannian n-manifolds N_k such that each U_k^ϵ is contained in a compact metric ball $B(p_k, R) \subset N_k$ for some $R > \epsilon > 0$. Assume that $\mathrm{Ric}(B(p_k, R)) \geq -(n-1)H$ for some $H > 0$. Then the absolute volume comparison implies that $\mathrm{vol}(U_k^\epsilon)$ is uniformly bounded above by $B^H(R)$. Now if $\mathrm{vol}(B(x_k, \epsilon/2))$ is uniformly bounded below, for *some* $x_k \in B(p_k, R-\epsilon)$, then the relative volume comparison ensures that $\mathrm{vol}(B(x,r))$ is uniformly bounded below for *any* $x \in B(p_k, R-\epsilon)$ and any $r < \epsilon/2$.

In particular, if each N_k is complete and $\sec(N_k) \geq -1$, then any sequence of compact domains U_k has uniformly bounded geometry provided $\mathrm{diam}(U_k) \leq D$ and there are points $x_k \in U_k$ with $\mathrm{vol}(B(x_k, r)) \geq v$.

Remark 3.7. Let U_k be a sequence of compact connected domains with smooth boundaries in Riemannian n-manifolds N_k such that U_k^ϵ is compact. Assume that $\mathrm{Ric}(U_k^\epsilon) \geq -(n-1)H$, for some $H > 0$. Then the following two conditions are equivalent

(i) $\mathrm{vol}(U_k^\epsilon) \leq V$, and $\mathrm{vol}(B(x,r)) \geq v$ for any $x \in U_k^{\epsilon/2}$

(ii) there is a point $x_0 \in U_k^{\epsilon/2}$ such that $\mathrm{vol}(B(x_0, r) \geq v$ and $\mathrm{diam}^{int}(U_k^{\epsilon/2}) \leq D$ where the diameter is taken with respect to the intrinsic distance induced by the Riemannian metric on $U_k^{\epsilon/2}$.

Indeed, let us show that (i)$\Rightarrow$(ii). Using the relative volume comparison we can make r and v slightly smaller so that $r < \epsilon/4$. Fix $\delta < r/100$ and find a path in $U_k^{\epsilon/2}$ of length between the numbers $\mathrm{diam}^{int}(U_k^{\epsilon/2})$ and $\mathrm{diam}^{int}(U_k^{\epsilon/2}) + \delta$. This path is almost length minimizing with the error $\leq \delta$. Hence, one can find $N = [\mathrm{diam}^{int}(U_k^{\epsilon/2})/3r]$ points on the path such that r-balls centered at the points are disjoint. Thus, $V \geq \mathrm{vol}(U_k^\epsilon) \geq Nv$ and we get a uniform bound on $\mathrm{diam}^{int}(U_k^{\epsilon/2})$.

Conversely, let us prove (ii)$\Rightarrow$(i). Fix $\delta < \epsilon/10$. First, show that there is a uniform lower bound for $\mathrm{vol}(B(x,\delta))$ for any $x \in U_k^{\epsilon/2}$. Take an arbitrary point $x \in U_k^\epsilon/2$. Since $\mathrm{diam}^{int}(U_k) \leq D$, there is a sequence of points $x_i \in U_k$, $i = 0, \ldots, N$ where $N = [D/\delta] + 1$ that starts at x_0, ends at $x_N = x$ and satisfies $d(x_i, x_{i+1}) \leq \delta$. Let $v_n(r, H)$ denote the volume of the ball of radius r in a complete simply connected n-dimensional space of constant sectional

curvature $= H$. Using induction on i and the relative volume comparison, one can show that for every i

$$\mathrm{vol}(B_\delta(p_i)) \geq \left(\frac{v_n(\delta, H)}{v_n(2\delta, H)}\right)^i v.$$

In particular, there is a uniform lower bound for $\mathrm{vol}(B(x,\delta))$.

Now fix a finite covering of $U_k^{\epsilon/2}$ by δ-balls. As before the relative volume comparison gives a uniform upper bound $N_{loc}(\delta)$ on the multiplicities of intersections in this covering (the argument involves only small balls so it works because the balls are far enough from the boundary.)

By the absolute volume comparison, the volume of each δ-ball is uniformly bounded above, hence a bound $\mathrm{vol}(U_k^\epsilon) \leq V$ would follow from a bound on the number of balls in the covering. Set $r_j = j\delta$, $j = 0, \ldots, m$ with $m = [D/\delta]+1$. Let N_j be the number of balls in the covering whose centers are in the r_j-ball around x_0 (as before the ball is taken with respect to the induced Riemannian metric on $U_k^{\epsilon/2}$). Since multiplicities are bounded by N_{loc}, for each j we have that $N_{j+1} \leq N_j + N_j^{N_{loc}(\delta)}$. This gives a uniform bound on the number of balls in the covering, and hence on $\mathrm{vol}(U_k^\epsilon)$.

4 Invariants of maps

Definition 4.1. *Let B be a topological space and $S(B)$ be a set. Given a smooth manifold N, denote by $C(B,N)$ the set of all continuous maps from B to N. Suppose that for any smooth N we have a map $\iota\colon C(B,N) \to S(B)$. Then we call ι an $S(B)$-valued invariant of maps of B if the two following conditions hold:*

(1) *Homotopic maps $f_1 : B \to N$ and $f_2\colon B \to N$ have the same invariant.*

(2) *Let $h\colon N \to L$ be a homeomorphism of N onto an open subset of L. Then, for any continuous map $f\colon B \to N$, the maps $f\colon B \to N$ and $h \circ f\colon B \to L$ have the same invariant.*

There is a variation of this definition for maps into oriented manifolds. Namely, we require that the target manifold is oriented and the homeomorphism h preserves orientation. In that case we say that ι is an *invariant of maps into oriented manifolds.*

Example 4.2. (Pontrjagin classes) As usual the total (rational) Pontrjagin class of a bundle ξ is denoted by $p(\xi)$. Given a continuous map of smooth

manifolds $f\colon B \to N$, set $p(f)$ to be the solution of $f^*p(TN) = p(TB) \cup p(f)$. (A total Pontrjagin class is a unit so there exists a unique solution.) The fact that $p(f)$ is an $H^*(B, \mathbb{Q})$-valued invariant follows from topological invariance of rational Pontrjagin classes [Nov66]. In case f is a smooth immersion, $p(f)$ is the the total Pontrjagin class of the normal bundle to f. Finally, note that Stiefel-Whitney classes are preserved by homeomorphism [Spa81] hence they also give rise to invariants of maps.

Remark 4.3. The isomorphism class of the pullback of the tangent bundle to N under f would be an invariant (for paracompact B) if we only require that invariants are preserved by diffeomorphisms. In general, homeomorphisms do not preserve tangent bundles. However, tangent bundle (and, in fact, any vector bundle over a finite cell complex) is recovered up to finitely many possibilities by the total Pontrjagin class and the Euler class of its orientable (1 or 2-fold) cover (see [Bel98] for a proof of this folklore result).

Example 4.4. (Intersection number in oriented n-manifolds.) Assume B is a compact space and fix two homology classes $\alpha \in H_m(B, \mathbb{Q})$ and $\beta \in H_{n-m}(B, \mathbb{Q})$. (Unless stated otherwise we always use singular (co)homology with rational coefficients.) Let $f\colon B \to N$ be a continuous map of a compact topological space B into an oriented n-manifold N. Set $I_{n,\alpha,\beta}(f)$ to be the intersection number of $f_*\alpha$ and $f_*\beta$ in N. It is easy to see that $I_{n,\alpha,\beta}$ is an $\mathbb{Q}$-valued invariant of maps into oriented manifolds.

Example 4.5. (Generalized Euler class) Let B be a closed oriented m-manifold and let $f\colon B \to N$ be a map of B into an oriented n-manifold N. Define the rational Euler class $\chi(f)$ by requiring that $\langle \chi(f), \alpha \rangle = I_{n,\alpha,[M]}$. This is a $H^{n-m}(B, \mathbb{Q})$-valued invariant for maps into oriented manifolds. If f is a smooth embedding, $\chi(f)$ is the Euler class of the normal bundle ν_f. Note that when the orientation is changed on B or N, the invariant $\chi(f)$ may change sign.

More generally, if B and N are not assumed to be orientable one can define a (generalized) Euler class of a continuous map as follows.

Recall that a smooth manifold L is orientable iff the first Stiefel-Whitney class $w_1(TL) \in H^1(L, \mathbb{Z}/2\mathbb{Z})$ vanishes. Note that

$$H^1(L, \mathbb{Z}/2\mathbb{Z}) \cong \mathrm{Hom}(H_1(L), \mathbb{Z}/2\mathbb{Z}) \cong \mathrm{Hom}(\pi_1(L), \mathbb{Z}/2\mathbb{Z}),$$

so elements of $H^1(L, \mathbb{Z}/2\mathbb{Z})$ correspond to subgroups of index ≤ 2 in $\pi_1(L)$ which are the kernels of homomorphisms in $\mathrm{Hom}(\pi_1(L), \mathbb{Z}/2\mathbb{Z})$.

Let K_f be the intersection of the subgroups corresponding to $w_1(B)$ and $f^*w_1(N)$. Let $\widetilde{B} \to B$ be a covering associated to K_f and let $\widetilde{N} \to N$ be a covering associated with $f_*(K_f)$. Then f lifts to a map $\tilde{f}\colon \widetilde{B} \to \widetilde{N}$ of orientable manifolds. Define the *generalized Euler class* $\tilde{\chi}(f)$ as a pair $(K_f, \pm\chi(\tilde{f}))$ (note that $\chi(\tilde{f})$ depends on the choice of orientations in $\widetilde{B}$ and $\widetilde{N}$, so it is only well-defined up to sign). It is easy to see that $\tilde{\chi}(f)$ is an invariant because homotopies and homeomorphisms lift to covering spaces, and because Stiefel-Whitney classes are topological invariants [Spa81].

Thus, for our purposes, $\tilde{\chi}(f)$ is a regular covering $\widetilde{B} \to B$ and two cohomology classes $\chi(\tilde{f})$, $-\chi(\tilde{f})$ in $H^{n-m}(\widetilde{B}, \mathbb{Q})$. For a map of orientable manifolds $f\colon B \to M$, $\tilde{\chi}(f) = (\pi_1(B), \pm\chi(f))$ so, up to sign, $\tilde{\chi}(f)$ generalizes $\chi(f)$.

If f is a smooth embedding of nonorientable manifolds with orientable normal bundle ν_f, then the Euler class of ν_f is taken to $\pm\chi(\tilde{f})$ by the map $H^{n-m}(B, \mathbb{Q}) \to H^{n-m}(\widetilde{B}, \mathbb{Q})$ induced by the covering.

Proposition 4.6. *Let $e_k\colon B \to N_k$ be a sequence of smooth embedding of a closed manifold B into manifolds N_k. Assume that invariants $p(e_k)$ and $\tilde{\chi}(e_k)$ are independent of k. Then the set of isomorphism classes of normal bundles ν_{e_k} is finite.*

Proof. It is well-known to experts that a vector bundle over a finite cell complex is recovered up to finitely many possibilities by the total Pontrjagin class and the Euler class of its orientable (1 or 2-fold) cover (see [Bel98] for a proof). We are now going to reduce to this result.

In what follows we use the notations of 4.5. Since $\tilde{\chi}(e_k) = (K_{e_k}, \pm\chi(\tilde{e}_k))$ is independent of k, there is a covering $\widetilde{B} \to B$ associated with $K_{e_k} \leq \pi_1(B)$ and, for each k, a covering $\widetilde{N}_k \to N_k$ associated with $e_{k*}(K_{e_k})$. The embedding e_k lifts to an embedding $\tilde{e}_k\colon \widetilde{B} \to \widetilde{N}_k$ of orientable manifolds.

Using that $H^1(B, \mathbb{Z}/2\mathbb{Z})$ is a finite group, we can partition ν_{e_k} into finitely many subsequences each having the same first Stiefel-Whitney class. It suffices to show that any such subsequence falls into finitely many isomorphism classes, so we can assume that $w_1(\nu_{e_k})$, and hence $e_k^*w_1(TN_k) = w_1(\nu_{e_k}) + w_1(TB)$, is independent of k. Let $\overline{B} \to B$ be a covering associated to the subgroup of $\pi_1(B)$ that corresponds to $w_1(\nu_{e_k})$. This subgroup lies in K_{e_k} so $\overline{B}$ is an intermediate covering space between $\widetilde{B}$ and B, that is, we have coverings $\tilde{q}\colon \widetilde{B} \to \overline{B}$ and $\bar{q}\colon \overline{B} \to B$. Also let $\overline{N}_k \to N_k$ be a covering associated to $e_{k*}\bar{q}_*(\pi_1(\overline{B}))$.

The embedding e_k lifts to an embedding $\bar{e}_k \colon \overline{B} \to \overline{N}$. Now the normal bundle $\nu_{\bar{e}_k}$ is orientable so its Euler class is well-defined (up to sign since there is no canonical choice of orientations). Note that $\tilde{q}^*$ takes the Euler class of $\nu_{\bar{e}_k}$ to $\pm\chi(\tilde{e}_k)$. It is a general fact that finite covers induce injective maps in rational cohomology. (The point is that the transfer map goes the other way, and precomposing the transfer with the homomorphism induced by the covering is multiplication by the order of the covering.) Also, the Pontrjagin class of $\nu_{\bar{e}_k}$ is $\bar{q}^*$-image of $p(e_k)$.

Thus, given $\chi(\tilde{e}_k)$ and $p(e_k)$, one can uniquely recover the (rational) Euler and Pontrjagin class of $\nu_{\bar{e}_k}$. As we mentioned above these classes determine ν_{e_k} up to finitely many possibilities. Therefore, ν_{e_k} are determined up to finitely many possibilities by $\tilde{\chi}(e_k)$ and $p(e_k)$ as desired. $\square$

Remark 4.7. The above proof actually gives a slightly more general result which will be useful in our applications. Namely, instead of assuming that e_k's are smooth embeddings, it suffices to assume that each e_k is a topological embedding such that $e_k(B)$ is a smooth submanifold of N_k. Then $e_k(B)$ has a normal bundle in N_k whose pullback via e_k is still denoted ν_{e_k}.

5 Main technical results

Proposition 5.1. *Let $f_k \colon M \to N_k$ be a sequence of continuous maps of a closed Riemannian manifold M into (possibly incomplete) Riemannian n-manifolds. Assume that, for each k, there exists a compact domain $U_k \supset f_k(M)$ such that $\{U_k\}$ has uniformly bounded geometry. Assume that either*

(i) *$\{f_k\}$ is almost equicontinuous, or*

(ii) *f_k is a homotopy equivalence with a homotopy inverse $g_k \colon N_k \to M$ such that $\{g_k\}$ is almost equicontinuous.*

Then, for any $S(M)$-valued invariant of maps ι, the subset $\{\iota(f_k)\}$ of $S(M)$ is finite.

Proof. Since $\{U_k\}$ has uniformly bounded geometry, there exists a metric space V, and homeomorphisms $h_k \colon V \to V_k$ of V onto compact domains $V_k \supset U_k$ such that both $\{h_k\}$ and $\{h_k^{-1}\}$ are almost equicontinuous.

If (i) holds, then $\{h_k^{-1} \circ f_k\}$ is an almost equicontinuous sequence of maps from M into V. Thus, 2.3 implies that the maps $h_k^{-1} \circ f_k$'s fall into finitely many homotopy classes. Now we are done by definition of an invariant since h_k^{-1}'s are homeomorphisms.

If (ii) holds, then $\{g_k \circ h_k\}$ is an almost equicontinuous sequence of maps from V into M. Again, by 2.3 there are only finitely many homotopy classes of maps among $g_k \circ h_k$. It suffices to show that, whenever $g_k \circ h_k$ is homotopic to $g_m \circ h_m$, the maps f_k and f_m have the same invariants. Let $G\colon V \times [0,1] \to M$ be a homotopy that connects $g_k \circ h_k$ and $g_m \circ h_m$. The homotopy $F\colon M \times [0,1] \to V$ defined as $F(x,t) = G(h_k^{-1}(f_k(x)), t)$ connects $g_m \circ h_m \circ h_k^{-1} \circ f_k$ with $g_k \circ h_k \circ h_k^{-1} \circ f_k = g_k \circ f_k \sim \mathrm{id}_M$. Thus, f_m is homotopic to

$$f_m \circ g_m \circ h_m \circ h_k^{-1} \circ f_k \sim \mathrm{id}_{N_m} \circ h_m \circ h_k^{-1} \circ f_k \sim h_m \circ h_k^{-1} \circ f_k.$$

Since $h_m \circ h_k^{-1}$ is a homeomorphism, f_k and f_m have the same invariants as desired. □

Remark 5.2. In the above theorem M can be chosen as in 2.4. Note that the spaces mentioned in 2.4 are homotopy equivalent to finite cell complexes [Wes77], in particular, characteristic classes determine a vector bundle over such a space up to finitely many possibilities.

Also, instead of assuming $\{g_k\}$ is almost equicontinuous, it is enough to assume that $g_k|_{V_k}$ is almost equicontinuous.

There is a version of the theorem for invariants of maps into oriented manifolds. First of all, by pulling back the orientation from V, one can *define* orientations on N_k so that h_k preserve orientations. In general, change of orientation on N may lead to an unknown change of an invariant of a map into N. However, if $\iota = I_{n,\alpha,\beta}$, then change of orientation on N may only lead to the sign change for the intersection number. Thus, for $\iota = I_{n,\alpha,\beta}$, the above theorem holds.

Corollary 5.3. *Let M be a closed Riemannian manifold and let $e_k\colon M \to N_k$ is a sequence of topological embeddings of M into Riemannian n-manifolds N_k such that $e_k(M) \subset N_k$ is a smooth submanifold. Assume that, for each k, e_k is homotopic to $f_k\colon M \to N_k$ and there exists a compact domain $U_k \supset f_k(M)$ such that $\{U_k\}$ has uniformly bounded geometry. Assume that either*

(i) *$\{f_k\}$ is almost equicontinuous, or*

(ii) *f_k is a homotopy equivalence with a homotopy inverse $g_k\colon N_k \to M$ such that $\{g_k\}$ is almost equicontinuous.*

Then the set of isomorphism classes of normal bundles ν_{e_k} is finite.

Proof. For any invariant ι, $\iota(f_k) = \iota(e_k)$. In particular this is true for the rational Pontrjagin class and generalized Euler class. The result now follows from 5.1 combined with 4.6, 4.7. □

Remark 5.4. Note that 5.3 also holds when e_k's are only immersions provided $\dim(N_k) - \dim(M)$ is either odd or $> \dim(M)$. Indeed, under these codimension assumptions the rational Euler class of ν_{e_k} vanishes while the total Pontrjagin class of ν_{e_k} is equal to $p(e_k)$.

Corollary 5.5. *Let $e_k\colon M_k \to N_k$ be a sequence of topological embeddings of closed Riemannian manifolds M_k into Riemannian n-manifolds N_k such that $e_k(M_k) \subset N_k$ is a smooth submanifold and $\{M_k\}$ has uniformly bounded geometry. Assume that e_k is homotopic to $f_k\colon M_k \to N_k$ and there exists a compact domain $U_k \supset f_k(M_k)$ such that $\{U_k\}$ has uniformly bounded geometry. Assume that either*

(i) $\{f_k\}$ *is almost equicontinuous, or*

(ii) f_k *is a homotopy equivalence with a homotopy inverse $g_k\colon N_k \to M_k$ such that $\{g_k\}$ is almost equicontinuous.*

Then the set of topological equivalence classes of normal bundles ν_{e_k} is finite.

Proof. Since M_k has bounded geometry, there exists M and homeomorphisms $h_k\colon M \to M_k$ such that both $\{h_k\}$ and $\{h_k^{-1}\}$ are almost equicontinuous. Note that $e_k(h_k(M)) = e_k(M_k)$ is a smooth submanifold of N_k. If $\{f_k\}$ is almost equicontinuous, then so is $\{f_k \circ h_k\}$. Similarly, if $\{g_k\}$ is almost equicontinuous, then so is $\{h_k^{-1} \circ g_k\}$. Thus, 5.3 implies that the set of isomorphism classes of normal bundles $\nu_{e_k \circ h_k}$ is finite. In particular, the set of topological equivalence classes of normal bundles ν_{e_k} is finite. □

Remark 5.6. For future applications we note that if h_k's are diffeomorphisms, then the conclusion of 5.5 can clearly be improved to "the set of smooth equivalence classes of normal bundles ν_{e_k} is finite."

6 Geometric applications

This section contains proofs of the various finiteness theorems that follow from section 5.

Corollary 6.1. *Let $e_\alpha\colon M \to N_\alpha$ be an almost equicontinuous family of smooth embeddings of a closed Riemannian manifold M into complete Riemannian n-manifolds N_α with $\sec(N_\alpha) \geq -1$. Assume that for each α there is a point $p_\alpha \in N_\alpha$ such that $\mathrm{vol}(B(p_\alpha, 1))$ is uniformly bounded below and $dist_{N_\alpha}(p_\alpha, e_\alpha(M))$ is uniformly bounded above. Then the set of isomorphism classes of normal bundles ν_{e_α} is finite.*

Proof. Since $\{e_\alpha\}$ is almost equicontinuous, $\operatorname{diam}(e_\alpha(M))$ is uniformly bounded above. The result now follows from 5.3 and section 3. $\square$

Proof of 1.1. Since $\operatorname{diam}(f(S)) \le D$ we can find a compact domain $U \supset f(S)$ with $\operatorname{diam}(U) \le 2D$. By results of the section 3, any family of such domains U has bounded geometry, hence the conclusion follows from 5.3. $\square$

Proof of 1.2. Let N_α be a family of nonnegatively curved manifolds satisfying conditions of 1.2. For any α let $S_\alpha \subset N_\alpha$ be a soul of N_α. First,we show that $\{S_\alpha\}$ has uniformly bounded geometry. By assumption S_α has lower volume bound. Lower sectional curvature bound follows because souls are totally geodesic. Since $\operatorname{diam}(f_\alpha(S_\alpha)) \le D$ and there is a distance-nonincreasing retraction of $r_\alpha \colon N_\alpha \to S_\alpha$ [Sha77], the diameter of $r_\alpha(f_\alpha(S_\alpha))$ is at most D. The map $r_\alpha \circ f_\alpha \colon S_\alpha \to S_\alpha$ is a homotopy equivalence, in particular, it has nonzero degree, hence it is onto. We conclude that $\operatorname{diam}(S_\alpha) \le D$. Thus, $\{S_\alpha\}$ has uniformly bounded geometry.

Since $\operatorname{diam}(f_\alpha(S_\alpha)) \le D$ we can find compact domains $U_\alpha \supset f_\alpha(S_\alpha)$ with $\operatorname{diam}(U_\alpha) \le 2D$. Again, $\{U_\alpha\}$ has uniformly bounded geometry, and the conclusion follows from 5.5. $\square$

We now prove a theorem that, in particular, implies 1.3.

Theorem 6.2. *Given n, K, D, r, v, there is a finite collection of vector bundles such that for any totally geodesic embedding of a closed Riemannian manifold M into a complete Riemannian manifold N, the normal bundle of M is topologically equivalent to a bundle of the collection provided* $\operatorname{diam}(M) \le D$, $\sec(N) \ge -1$, *and there exist positive r, v, and a point $p \in e(M)$ such that* $\operatorname{vol}_N B(p,r) \ge v$.

Proof. Start with an arbitrary family of totally geodesic embeddings $e_\alpha \colon M_\alpha \hookrightarrow N_\alpha$ as above. First, we show that $\{M_\alpha\}$ has uniformly bounded geometry where M_α is equipped with the induced Riemannian metric. By a result of Karcher-Heinze [HK78] $\operatorname{vol}_{N_\alpha} B(p_\alpha, r) \ge v$ implies a lower volume bound on M_α. Since M_α is totally geodesic $\sec(M_\alpha) \ge -1$, and by assumption $\operatorname{diam}(M_\alpha) \le D$. Thus, Perelman's stability theorem implies that $\{M_\alpha\}$ has uniformly bounded geometry (see 3.5)

Note that $\operatorname{diam}(e_\alpha(M_\alpha)) \le \operatorname{diam}(M_\alpha) \le D$, hence, 3.5 implies that there is a compact domain $W_\alpha \supset e_\alpha(M_\alpha)$ such that $\{W_\alpha\}$ has uniformly bounded geometry. The result now follows from 5.5(ii) because totally geodesic embeddings $\{e_\alpha\}$ are 1-Lipschitz, in particular, $\{e_\alpha\}$ is equicontinuous. $\square$

The same proof gives the following.

Theorem 6.3. *Given positive n, H, D, i_0, and ϵ, there is a finite collection of vector bundles such that for any totally geodesic embeddings of a closed Riemannian manifold M into a complete Riemannian manifold N, the normal bundle of M is topologically equivalent to a bundle of the collection provided* $\mathrm{diam}(M) \le D$, $\mathrm{Ric}(N) \ge -1$, *and* $\mathrm{inj}(x) \ge i_0$ *for any x in the ϵ-neighborhood of image of M.* □

Remark 6.4. There are obvious "fixed base" modifications of 6.2 and 6.3.

Corollary 6.5. *Given positive D, r, v, n, and a closed manifold M with $\oplus_{i>0}H^{4i}(M,\mathbb{Q}) = 0$, there exists a finite collection of vector bundles over M such that, for any open complete Riemannian n-manifold N with* $\sec(N) \ge 0$ *and a soul $S \subset N$, the normal bundle to S is topologically equivalent to a bundle of the collection provided S is homeomorphic to M, and the inclusion $S \hookrightarrow N$ is homotopic to a map f such that* $\mathrm{diam}(f(S)) \le D$ *and* $\mathrm{vol}_N(B(p,r)) \ge v$ *for some $p \in f(S)$.*

Proof. First note that, up to topological equivalence, only finitely many of the bundles ν_S can have zero Euler class. (Otherwise, there is a sequence of pairwise topologically inequivalent bundles ν_{S_k} with zero Euler class. Use homeomorphisms $M \to S_k$ to pull the bundles back to M. These pullback bundles clearly have zero Euler class as well as zero rational Pontrjagin classes since $\oplus_{i>0}H^{4i}(M,\mathbb{Q}) = 0$. Thus the bundles belong to finitely many isomorphism classes which implies that ν_{S_k} belong to finitely many topological equivalence classes.) Now if the Euler class of the normal bundle to S is nonzero, then $f(S) \cap S \neq \emptyset$, hence the distance from p to S is $\le D$, and the result follows from 6.2. □

Proof of 1.4. Let N^n and $p \in S \subset N$ be chosen to satisfy the assumptions. According to 1.1 we only have to show is that under our assumptions we have a uniform lower bound on $\mathrm{vol}B(p,r)$. Let $r_0 = \min\{r/2, \pi/(2\sqrt{K})\}$. Let $l = codim S - 1$ and (S^l_K, g_{can}) be a round sphere of constant curvature K and $\bar{p}$ be any point on this sphere. Consider the exponential map $\exp_K \colon T_{\bar{p}}S^l_K \to S^l_K$. Denote by $v(l,K,t)$ the volume of the ball of radius t centered at $\bar{p}$.

First of all, notice that by the triangle inequality $B(p,r)$ contains a tubular neighborhood $U(p,r_0)$ consisting of all points $x \in N$ such that $d(x,S) \le$

r_0 and $d(p, Sh(x)) \le r_0$. Here Sh stands for the Sharafutdinov retraction $Sh\colon N \to S$. For any $x \in S$ denote

$$B^{\perp}(x,t) = \{y \in N | d(y,S) \le t \text{ and } Sh(y) = x\}.$$

Since Sharafutdinov retraction is a C^1-Riemannian submersion [Per94] we can apply Fubini's Theorem to see that

$$volU(p,r_0) = \int_{B_S(p,r_0)} \mathrm{vol}B^{\perp}(x,r_0)\mathrm{dvol}(x) \tag{1}$$

Here $B_S(p,r_0)$ stands for the ball of radius r_0 around p in S. It suffices to show that for each $x \in B_S(p,r_0)$ we have $\mathrm{vol}B^{\perp}(x,r_0) \ge v(l,K,r_0)$. (Indeed, it would imply that $\mathrm{vol}(B(p,r)) \ge \mathrm{vol}(U(p,r_0)) \ge \mathrm{vol}(B_S(p,r_0)) \cdot v(l,K,r_0)$. Finally, by volume comparison, $\mathrm{vol}(B_S(p,r_0))$ is bounded below in terms of D and v and we are done.)

Fix an $x \in B_S(p,r_0)$ and consider the normal exponential map $\exp_x^{\perp}\colon T_x^{\perp}S \to N$. It follows from [Per94] that this map sends the ball $B_{T_x^{\perp}}(0,r_0)$ onto $B^{\perp}(x,r_0)$. Choose a linear isometry between $T_x^{\perp}$ and $T_{\bar{p}}S_K^l$ and use it to equip $B_{T_x^{\perp}}(0,r_0)$ with the metric g_K of constant curvature K. Let g_x be the induced Riemannian metric on the Sharafutdinov fiber over x. To finish the proof it is enough to establish the following lemma saying that "reverse Toponogov comparison" holds on $B^{\perp}(x,r_0)$.

Lemma 6.6. *The surjection* $\exp_x^{\perp}\colon (B_{T_x^{\perp}}(0,r_0), g_K) \to (B^{\perp}(x,r_0), g_x)$ *is a distance nondecreasing diffeomorphism.*

Let v be a unit vector in $T_x^{\perp}$ and $\gamma(t) = \exp(tv)$ be the normal geodesic in direction v. We now show that, for any $t \le r_0$ and any $X \in T_{\gamma(t)}$ with $|X| = 1$ and $\langle X, \gamma'(t)\rangle$, we have that $K(X,\gamma'(t)) \le K$. Write $X = X^h + X^v$ as a sum of its horizontal and vertical components. Then

$K(X,\gamma'(t)) = \langle R(\gamma'(t), X^h + X^v)\gamma'(t), X^h + X^v\rangle = \langle R(\gamma'(t), X^v)\gamma'(t), X^v\rangle + \langle R(\gamma'(t), X^h)\gamma'(t), X^h\rangle + \langle R(\gamma'(t), X^h)\gamma'(t), X^v\rangle + \langle R(\gamma'(t), X^v)\gamma'(t), X^h\rangle$.

The first term in the right hand side is $\le K$ by assumption and also because $|X^v| \le |X| = 1$. By [Per94] $R(\gamma'(t), X^h)\gamma'(t) = 0$ and therefore

$$\langle R(\gamma'(t), X^h)\gamma'(t), X^v\rangle = \langle R(\gamma'(t), X^h)\gamma'(t), X^h\rangle = 0.$$

By the symmetry of the curvature tensor the forth term is equal to the third one and hence is also equal to 0. Thus $K(X, \gamma'(t) = \langle R(\gamma'(t), X^h)\gamma'(t), X^h\rangle \le K$.

Now since $r_0 < \pi/\sqrt{K}$ and all the two planes along $\gamma(t)$ containing $\gamma'(t)$ have curvature $\le K$ we can apply the Rauch comparison theorem to conclude that the differential of $\exp_x^{\perp}$ does not decrease the lengths of tangent vectors and

thus $\exp_x^\perp\colon (B_{T_x^\perp}(0,r_0),g_K)\to(B^\perp(x,r_0),g_x)$ is a local diffeomorphism that does not decrease lengths of curves. It remains to show that this map is $1-1$.

Suppose not. Then the injectivity radius r_x of $B^\perp(x,r_0)$ at x is strictly less than r_0. Let $v,u\in T_x^\perp$ be such that $|u|=|v|=r_x$ and $\exp(v)=\exp(u)$. Denote $q=\exp(v)=\exp(u)$. Notice that geodesics $\gamma_v(t)=\exp(tv)$ and $\gamma_u(t)=\exp(tu)\colon [0,1]\to N$ connecting x and q are obviously distance minimizing. By [CE75, Lemma 5.6.5] these geodesics must form a geodesic loop at x (i.e $\gamma_u'(1)=-\gamma_v'(1)$). This is impossible since according to [CG72] the soul S is totally convex and x lies in S. □

Corollary 6.7. *Given positive D, n, v, r, and K, there exists a finite family of vector bundles such that, for any complete open Riemannian n-manifold N with $\sec(N)\ge 0$ and a soul $S\subset N$, the normal bundle to S is topologically equivalent to a bundle of the collection provided $\operatorname{diam}(S)\le D$, $\operatorname{vol}(S)\ge v$ and there is a point $p\in S$ such that all the vertical curvatures at the points of $B(p,r)$ are bounded above by K.*

Proof. The proof of 1.4 gives a uniform lower bound on $\operatorname{vol}B(p,r)$ so the result follows from 1.2. □

Corollary 6.8. *Given positive D, r, v, n, K, and a closed Riemannian manifold M with $\sec(M)\in[-1,0]$, there exists a finite collection of vector bundles over M such that, for any totally geodesic embedding $e\colon M\to N$ of M into an open complete Riemannian n-manifold N with $\sec(N)\le 0$, the normal bundle ν_e is isomorphic to a bundle of the collection provided $\operatorname{vol}(M)\ge v$ and e is a homotopy equivalence homotopic to a map f with $\operatorname{diam}(f(M))\le D$ such that the sectional curvature at any point of the r-neighborhood of $f(M)$ is $\ge -K$.*

Proof. Start with an arbitrary family of totally geodesic embeddings $e_\alpha\colon M\hookrightarrow N_\alpha$ as above. First, note that for any α the injectivity radius of N_α satisfies $\operatorname{inj}(N_\alpha)\ge\operatorname{inj}(M)=\operatorname{inj}(e_\alpha(M))$. If not, there is a point $p\in N_\alpha$ with $\operatorname{inj}_{N_\alpha}(p)<\operatorname{inj}(M)$. Since N_α and M has nonpositive sectional curvatures, the injectivity radius at any point is half the length of the shortest geodesic loop at this point [CE75, Lemma 5.6.5]. Take a geodesic loop at p of length $<\operatorname{inj}(M)/2$ and project it to $e_\alpha(M)$ by the closest point retraction $r_\alpha\colon N_\alpha\to e_\alpha(M)$ [BGS85]. The retraction is a distance-nonincreasing homotopy equivalence. (In fact, since $\sec(N_\alpha)\le 0$, the normal exponential map identifies N_α with the normal bundle to $e_\alpha(M)$ where r_α corresponds to the bundle projection.) Thus, we get a homotopically nontrivial curve of length $<\operatorname{inj}(M)/2$ in $e_\alpha(M)$. Since e_α is

an isometric embedding it preserves lengths of curves. Therefore, we obtain a homotopically nontrivial curve of length $< \mathrm{inj}(M)/2$ in M which is impossible because loops of length $< \mathrm{inj}(M)/2$ lift to loops in the universal cover.

Find compact domains $U_\alpha \supset f_\alpha(M)$ that lie in the r-neighborhood of $f_\alpha(M)$. Since $\mathrm{diam}(f_\alpha(M)) \le D$, the diameter of U_α is $\le D + 2r$. Also $\sec(U_\alpha) \in [-K, 0]$ and $\mathrm{inj}(U_\alpha) \ge \mathrm{inj}(M)$, hence 3.1 implies that $\{U_\alpha\}$ has bounded geometry, and the conclusion follows from 5.3. □

Proof of 1.5. Since $\sec(M_\alpha)$ is bounded in absolute value, lower bound on volume implies a lower bound on the injectivity radius. Thus, the result follows from 5.5 exactly as in the proof of 6.8. □

Remark 6.9. In the statement of 1.5 one can replace "$\mathrm{vol}(M) \ge v$" by "$\pi_1(M)$ has no normal virtually abelian subgroups", or equivalently, by the universal cover of M has no Euclidean de Rham factor (see e.g. [Fuk88, pp395–396]).

Remark 6.10. Given a closed Riemannian manifold M, there are only finitely many isomorphism classes of normal bundles of isometric immersions $f\colon M \to N$ into Riemannian manifolds N such that $|\sec(N)|$ and the second fundamental form of f are uniformly bounded. Indeed, these bounds imply a uniform bound on the curvature form of the normal bundle to ν_f. Then by Chern-Weil theory, we get bounds on Euler and Pontrjagin classes which determine a vector bundle up to finitely many possibilities. An alternative proof was recently found by K. Tapp [Tap99]. Instead of getting bounds on the characteristic classes, Tapp estimates the number of homotopy classes of maps into the classifying space.

A Vector bundles with diffeomorphic total spaces

The purpose of this appendix is to discuss to what extent a vector bundle is determined by its total space. We got interested in this problem when we noticed that under the assumptions of the corollary 1.3, the homeomorphism finiteness for the total spaces can be easily obtained from the parametrized version of Perelman's stability theorem [Per91].

Let η_k be an infinite sequence of vector bundles over a closed smooth manifold M such that the total spaces $E(\eta_k)$ are homeomorphic. Assume that the natural homomorphism $\mathrm{Homeo}(M) \to \mathcal{E}(M)$ of the homeomorphism group of M into the group of (free) homotopy classes of self-homotopy equivalences of M

has finite cokernel. Then the homeomorphism type of the total space determines (the topological equivalence class of) a vector bundle, up to a finite ambiguity. (Indeed, the homeomorphism $E(\eta_i) \to E(\eta_1)$ induces a self-homotopy equivalence g_i of M. Passing to a subsequence, we can assume that $g_i^{-1} \circ g_j$ is homotopic to a homeomorphism for any i, j. Let s_i be the zero section of η_i. Then the maps s_j and $s_i \circ g_i^{-1} \circ g_j$ have equal invariants in the sense of section 4. Hence, for a fixed j, 4.6 implies that the bundles $(g_i^{-1} \circ g_j)^{\#}\eta_i$ fall into finitely many isomorphism classes. Therefore, the bundles η_k fall into finitely many topological equivalence classes.)

For example, if the group $\mathcal{E}(M)$ is finite, then the homeomorphism type of the total space determines (the isomorphism class of) a vector bundle, up to a finite ambiguity. In fact, since a vector bundle is determined, up to a finite ambiguity, by its characteristic classes, it suffices to assume that the natural action of $\mathcal{E}(M)$ on the cohomology groups that contain these classes is an action of a finite group. For instance, up to isomorphism, there are only finitely many vector bundles over CP^n with homeomorphic total spaces because $\mathcal{E}(CP_n)$ is finite (see [Rut97, ch12, 18.3] for many more examples, also see [SW99] for the case when M is a sphere).

Also if the image of the diffeomorphism group $\mathrm{Diffeo}(M)$ in $\mathcal{E}(M)$ has finite index, then the homeomorphism type of the total space determines (the smooth equivalence class of) a vector bundle, up to finite ambiguity. If $dim(M) \geq 6$ and $\pi_1(M) = 1$, then the image of $\mathcal{D}(M)$ in $\mathcal{E}(M)$ is commensurable to the isotropy subgroup of the total rational Pontrjagin class of TM [Sul77, pp322–323]. In particular, the image of $\mathcal{D}(M)$ has finite index in $\mathcal{E}(M)$ when $p(TM) = 1$. For instance, up to smooth equivalence, there are only finitely many vector bundles with homeomorphic total spaces over the direct product of finitely many spheres.

Now we give an example of an infinite sequence of pairwise topologically inequivalent vector bundles with diffeomorphic total spaces. The base manifold will be homotopy equivalent to $S^3 \times S^3 \times S^5$. We are thankful to Shmuel Weinberger for providing a key idea for this example (as usual, the authors assume all responsibility for possible mistakes).

Example A.1. Consider a manifold $N = S^3 \times S^3 \times S^5$ and let t be a nonzero element of $H^8(N, \mathbb{Z})$. One can always find a positive integer q and a vector bundle τ over N of rank $\geq \dim(N)$ such that $qt = p_2(\tau) \in H^{12}(N, \mathbb{Z})$ where p_2 is the second integral Pontrjagin class. (Indeed, the Pontrjagin character ph defines an isomorphism of vector spaces

$$ph\colon\ \widetilde{K}(N) \otimes \mathbb{Q} \to \oplus_{i>0} H^{4i}(N, \mathbb{Q}).$$

Consider a finite sum $\sum_i [\eta_i] \otimes p_i/q_i$, where p_i, q_i are integers and η_i are bundles over N, such that $ph(\sum_i [\eta_i] \otimes p_i/q_i) = t \otimes 1$. Reducing to a common denominator and using that $p_i[\eta_i]$ is the class of p_i-fold Whitney power of η_i, we get $t \otimes 1 = ph(\xi)/q'$ for some positive integer q' and a bundle ξ over N; one can choose the rank of ξ to be any number ≥ 15. Since the first Pontrjagin class lives in the zero group, the formula for the ph_2, the part of the Pontrjagin character that lives in 8th cohomology, reduces to $ph_2 = -p_2/6$ and we are done.)

Replacing τ with its Whitney power, we can assume that τ is fiber homotopy equivalent to the trivial bundle TN. (Note that $p_2(\tau)$ is still an integral multiple of t.) Since $dim(N)$ is odd and ≥ 5, and N is simply connected a result of Browder-Novikov [Bro72, II.3.1] implies that there is a closed smooth manifold M and a homotopy equivalence $f\colon M \to N$ such that $f^\# \tau$ is stably isomorphic to TM.

It follows from [Sie70] that any automorphism of $H_3(S^3 \times S^3, \mathbb{Z}) \cong \mathbb{Z}^2$ is induced by a self-homotopy equivalence. Since the inclusion $S^3 \times S^3 \to N$ induces an isomorphism of the third integral homology groups, any automorphism of $H_3(N, \mathbb{Z})$ is induced by a self-homotopy equivalence. The same is therefore true for M. Furthermore, any automorphism of $H^8(M, \mathbb{Z})$ is induced by a self-homotopy equivalence. (Indeed, start with $\phi \in \mathrm{Aut}(H^8(M, \mathbb{Z}))$ and conjugate to by the Poincarë duality to get an automorphism ϕ' of $H_3(N, \mathbb{Z})$. If f is a self-homotopy equivalence of M inducing ϕ', then ϕ can be identified with the transfer map for f. The transfer map is the inverse to f^*, hence f^{-1} induces ϕ.)

Note that $\mathrm{Aut}(H^8(M, \mathbb{Z})) \cong GL(2, \mathbb{Z})$. Recall that any vector $(a, b) \in \mathbb{Z}^2$ is $GL(2, \mathbb{Z})$-equivalent to $(k, 0)$ where $k = \gcd(a, b)$. The vectors $v_m = (k, km)$ are $GL(2, \mathbb{Z})$-equivalent to $(k, 0)$ and lie in different orbits of the $GL(2, \mathbb{Z})$-stabilizer of $(k, 0)$.

Thus one can find an infinite sequence of elements $w_m \in H^8(M, \mathbb{Z}) \cong \mathbb{Z}^2$ that lie in different orbits of the stabilizer of $p_2(TM)$ in the group $GL(2, \mathbb{Z})$ and are $GL(2, \mathbb{Z})$-equivalent to $p_2(TM)$. Find self-homotopy equivalences g_m of M such that $w_m = g_m^* p_2(TM)$; let $g_1 = \mathrm{id}$. Let η_m be a bundle over M with $[\eta_m] = [g_m^\# TM] - [TM]$; one can choose the rank of η_m to be any number ≥ 15.

Now if η_i is topologically equivalent to η_j, then there is a selfhomeomorphism h of M that takes $p_2(\eta_i) = p_2(g_i^\# TM) - p_2(TM)$ to $p_2(\eta_j) = p_2(g_j^\# TM) - p_2(TM)$ (p_2 is additive because $p_1 = 0$). Since homeomorphisms preserve rational Pontrjagin classes, we get $h^* g_i^* p_2(TM) = g_j^* p_2(TM)$, up to elements of finite order. The group $H^8(M, \mathbb{Z})$ is torsion free, hence the above equality

hold exactly. So $h^*w_i = w_j$ which contradicts the definition of w_m. Thus, η_m are pairwise topologically inequivalent.

The map g_m induces a homotopy equivalence from the total space $E(\eta_m)$ to $E(\eta_1)$. Now $[TE(\eta_m)] = [\eta_m] + [TM] = [g_m^{\#}TM] - [TM] + [TM] = [g_m^{\#}TM]$. Thus, the homotopy equivalence $E(\eta_m) \to E(\eta_1)$ is tangential and hence is homotopic to a diffeomorphism [LS69, pp226-228].

Remark A.2. We now briefly describe a generalization of the above example. First of all, instead of $S^3 \times S^3$, one can start with a product P of an arbitrary number of 1-connected homology m-spheres with odd $m > 1$. (Any odd-dimensional sphere is rationally homotopy equivalent to a $K(\mathbb{Z}, m)$ space, so the product P is rationally homotopy equivalent $K(\mathbb{Z}^k, m)$ where k is the number of factors. This implies that the natural action $\mathcal{E}(P) \to \mathrm{Aut}(H_m(P, \mathbb{Z})) \cong GL(k, \mathbb{Z})$ has finite cokernel. Thus, "almost" every automorphism is induced by a homotopy equivalence which turns out to be enough for us.)

By taking a product with a suitable 1-connected manifold S (which was S^5 in the example) we can shift dimensions so that $\dim(P \times S)$ is odd and

$$\mathcal{E}(P \times S) \to \mathrm{Aut}(H^{4i}(P \times S, \mathbb{Z})) \cong GL(k, \mathbb{Z})$$

has finite cokernel for some i. Using [Bro72, I.3.1], we replace $P \times S$ by a homotopy equivalent manifold M with a nonzero Pontrjagin class p_i. (The freedom in the choice of τ gives infinitely many possibilities for the homeomorphism type of M.) Now it not hard to cook up an infinite sequence of pairwise topologically inequivalent bundles η_j over M with diffeomorphic total spaces. One can also get nonsimply connected examples by taking products of $E(\eta_j) \times L$ where L is a suitable closed manifold.

References

[AC92] M. T. Anderson and J. Cheeger, *C^α-compactness for manifolds with Ricci curvature and injectivity radius bounded below*, J. Differential Geom. **35** (1992), no. 2, 265–281.

[And87] M. T. Anderson, *Metrics of negative curvature on vector bundles*, Proc. Amer. Math. Soc. **99** (1987), no. 2, 357–363.

[And89] M. T. Anderson, *Ricci curvature bounds and Einstein metrics on compact manifolds*, J. Amer. Math. Soc. 2 (1989), no. 3, 455–490 **2** (1989), no. 3, 455–490.

[Bel98] I. Belegradek, *Pinching, Pontrjagin classes, and negatively curved vector bundles*, preprint, 1998.

[BGS85] W. Ballman, M. Gromov, and V. Schroeder, *Manifolds of nonpositive curvature*, Birkhäuser, Boston, 1985.

[BK00a] I. Belegradek and V. Kapovitch, *Obstructions to nonnegative curvature and rational homotopy theory*, in preparation, 2000.

[BK00b] I. Belegradek and V. Kapovitch, *Topological obstructions to nonnegative curvature*, preprint, 2000.

[Bor67] K. Borsuk, *Theory of retracts*, Państwowe Wydawnictwo Naukowe, Warsaw, 1967, Monografie Matematyczne, Tom 44.

[Bro72] W. Browder, *Surgery on simply-connected manifolds*, Springer-Verlag, New York, 1972, Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 65.

[CE75] J. Cheeger and D. Ebin, *Comparison theorems in Riemannian geometry*, North-Holland Publishing Co., Amsterdam, 1975, North-Holland Mathematical Library, Vol. 9.

[CG72] J. Cheeger and D. Gromoll, *On the structure of complete manifolds of nonnegative curvature*, Ann. of Math. **96** (1972), 413–443.

[Che73] J. Cheeger, *Some examples of manifolds of nonnegative curvature*, J. Differential Geom **8** (1973), 623–628.

[Cro84] C. B. Croke, *Curvature free volume estimates*, Invent. Math. **76** (1984), no. 3, 515–521.

[Fuk88] K. Fukaya, *A compactness theorem of a set of aspherical riemannian orbifolds*, A féte of topology, Academic Press, Boston, MA, 1988, pp. 391–413.

[Gro81] M. Gromov, *Structures métriques pour les variétés riemanniennes*, CEDIC, 1981, Edited by J. Lafontaine and P. Pansu.

[GW98] L. Guijarro and G. Walschap, *Twisting and nonnegative curvature metrics on vector bundles over the round sphere*, preprint, 1998.

[GZ] K. Grove and W. Ziller, *Bundles with nonnegative curvature*, in preparation.

[GZ99] K. Grove and W. Ziller, *Curvature and symmetry of Milnor spheres*, preprint, 1999.

[Hir95] T. Hiroshima, *C^α compactness theorem for Riemannian manifolds with bounds on diameter, injectivity radius, and some integral norms of Ricci curvature*, Indiana Univ. Math. J. **44** (1995), no. 2, 397–411.

[HK78] E. Heintze and H. Karcher, *A general comparison theorem with applications to volume estimates for submanifolds*, Ann. Sci. École Norm. Sup. (4) **11** (1978), no. 4, 451–470.

[HM74] M. Hirsch and B. Mazur, *Smoothings of piecewise linear manifolds*, Princeton University Press, Princeton, N. J., 1974, Annals of Mathematics Studies, No. 80.

[KS77] R. Kirby and L. Siebenmann, *Foundational essays on topological manifolds, smoothings, and triangulations*, Princeton University Press, Princeton, N.J., 1977, With notes by John Milnor and Michael Atiyah, Annals of Mathematics Studies, No. 88.

[LS69] W. B. R. Lickorish and L. S. Siebenmann, *Regular neighbourhoods and the stable range*, Trans. Amer. Math. Soc **139** (1969), 207–230.

[Moi77] E. E. Moise, *Geometric topology in dimensions* 2 *and* 3, Graduate Texts in Mathematics, vol. 47, Springer-Verlag, New York-Heidelberg, 1977.

[MSY93] N. Mok, Y. Siu, and S-K. Yeung, *Geometric superrigidity*, Invent. Math. **113** (1993), no. 1, 57–83.

[Mun60] J. Munkres, *Obstructions to the smoothing of piecewise-differentiable homeomorphisms*, Ann. of Math. **72** (1960), 521–554.

[Nov66] S. P. Novikov, *On manifolds with free abelian fundamental group and their application*, Izv. Akad. Nauk SSSR Ser. Mat. **30** (1966), 207–246, (Russian).

[ÖW94] M. Özaydin and G. Walschap, *Vector bundles with no soul*, Proc. Amer. Math. Soc. **120** (1994), no. 2, 565–567.

[Per91] G. Perelman, *Alexandrov spaces with curvatures bounded from below* II, preprint, 1991.

[Per94] G. Perelman, *Proof of the soul conjecture of Cheeger and Gromoll*, J. Differential Geom. **40** (1994), 209–212.

[Pet93] P. Petersen, *Gromov-Hausdorff convergence of metric spaces*, Differential geometry: Riemannian geometry (Los Angeles, CA, 1990), Amer. Math. Soc., Providence, RI, 1993, pp. 489–504.

[Pet97] P. Petersen, *Convergence theorems in Riemannian geometry*, Comparison geometry (Berkeley, CA, 1993–94), Cambridge Univ. Press, Cambridge, 1997, pp. 167–202.

[Rig78] A. Rigas, *Geodesic spheres as generators of the homotopy groups of* O*,* BO, J. Differential Geom. **13** (1978), no. 4, 527–545 (1979).

[Rut97] J. W. Rutter, *Spaces of homotopy self-equivalences. A survey*, Lecture Notes in Mathematics 1662, Springer-Verlag, Berlin, 1997.

[Sha77] V. A. Sharafutdinov, *The Pogorelov-Klingerberg theorem for manifolds homeomorphic to* R^n, Sib. Math. Zh. **18** (1977), 915–925, Russian.

[Sie70] A. J. Sieradski, *Twisted self-homotopy equivalences*, Pacific J. Math. **34** (1970), 789–802.

[Spa81] E. Spanier, *Algebraic topology*, Springer-Verlag, New York–Berlin, 1981, corrected reprint.

[Sul77] D. Sullivan, *Infinitesimal computations in topology*, Inst. Hautes Études Sci. Publ. Math. (1977), no. 47, 269–331.

[SW99] R. De Sapio and G. Walschap, *Diffeomorphism of total spaces and equivalence of bundles*, preprint, 1999.

[SY97] R. Schoen and S.T. Yau, *Lectures on harmonic maps*, International Press, 1997.

[Tap99] K. Tapp, *Finiteness theorems for bundles*, preprint, 1999.

[Wes77] J. West, *Mapping Hilbert cube manifolds to ANR's: a solution of a conjecture of Borsuk*, Ann. Math. (2) **106** (1977), no. 1, 1–18.

[Yan95] D. Yang, *On complete metrics of nonnegative curvature on 2-plane bundles*, Pacific J. Math. **171** (1995), no. 2, 569–583.